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ON THE STABILITY OF A TIME DEPENDENT BOUNDARY LAYER.

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ABSTRACT

The aim of this article is to determine the stability characteristics of a Rayleigh layer, which is known to occur when the fluid above a flat plate has a velocity imparted to it (parallel to the plate). This situation is intrinsically unsteady, however as a first approximation we consider the instantaneous stability of the flow. The Orr-Sommerfeld equation is found to govern fixed downstream wavelength linear perturbations to the basic flow profile. By the solution of this equation we can determine the Reynolds numbers at which the flow is neutrally stable; this quasi-steady approach is only formally applicable for infinite Reynolds numbers. We shall consider the large Reynolds number limit of the original problem and use a three deck mentality to determine the form of the modes. The results of the two calculations are compared, and the linear large Reynolds number analysis is extended to consider the effect of weak nonlinearity in order to determine whether the system is sub or super critical.

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§1 Introduction

The main thrust of boundary layer stability calculations have involved spatially evolving problems. Relatively little work has gone into the stability of temporally developing layers, one such situation is the topic of this paper, that of an impulsive flow over a flat plate.

The calculation to determine the linear stability of the system considered here is initially attempted using a quasi-steady approach where the flow is frozen at a certain time. This yields an Orr-Sommerfeld equation, the solutions of which were discussed for a variety of flows by Tollmien (1929) and Schlichting (1933), using homogeneous assumptions in the temporal and downstream coordinates. Unfortunately the true system governing what we shall refer to as Tollmien-Schlichting waves is not uniform for a general boundary layer. An attempt to remedy this deficiency was proposed for the Taylor-Görtler problem by Smith (1955). These ideas were extended by Bouthier (1973) and Gaster (1974), where extra terms were added to appreciate for the boundary layer's growth; although it was noted by Gaster that this only amounted to a successive approximation technique. It was in the work of Smith (1979 a) that a more formal asymptotic treatment of the problem was proffered for large Reynolds numbers. Smith used a triple deck structure to describe the linear development of fixed frequency Tollmien-Schlichting waves within a growing boundary layer, specifically a Blasius layer. It was found that the boundary layer growth altered the higher order terms in the expansions of various wave characteristics within this structure. The inclusion of these terms may be stabilising or destabilising depending on which quantity is taken to be representative of the perturbations intensity.

Some studies have been made of the stability characteristics of unsteady boundary layers, although most of the cases have involved temporal periodicity. In these problems either a quasi-steady approach or Floquet theory can be used. In Kerczek & Davis (1974) the linear stability of a Stokes layer on a flat plate is discussed, in this problem, waves that develop quickly compared to the temporal scale of the basic state are considered, as is the case in Cowley (1986). In Hall (1975) a study was made of unsteady flows about cylinders, and in Seminara & Hall (1976) the Taylor problem is considered where the inner cylinder is taken to oscillate periodically with time. The work of Di Prima & Stuart (1975) regards the Taylor problem for eccentric cylinders (which corresponds to a journal bearing). In Otto (1993) the stability of a boundary

layer at an infinite cylinder is discussed. The cylinder starts to spin up and vortices are generated by an axisymmetric element of wall roughness; these may lead to some kind of transition. In Duck (1990) the unsteady triple deck equations were solved for the flow over a small hump on a flat plate.

In the current problem the basic state depends on the normal and temporal coordinates, so it is natural to solve an unsteady problem. This situation is fully parallel so there is no need to make any assumptions about the spatial evolution of the basic state. After a time t_0 , we may define the boundary layer Reynolds number as,

$$R_{e0} = \frac{U_\infty (\nu t_0)^{\frac{1}{2}}}{\nu} = \frac{U_\infty t_0^{\frac{1}{2}}}{\nu^{\frac{1}{2}}},$$

based on the viscous layer's thickness. The velocity imparted to the fluid is U_∞ , and its kinematic viscosity is ν . In the viscous layer (which may be shown to have thickness $(\nu t)^{\frac{1}{2}}$), the basic state is found to be similar in time and is given by an error function. This is accreditable to the large Reynolds number assumption; at finite Reynolds numbers the unsteady equations need to be solved. We wish to consider the susceptibility of this profile to Orr-Sommerfeld modes that evolve on short temporal and spatial scales. As mentioned previously the approximation is made that we shall 'freeze' the basic profile. In an infinite Reynolds number flow this is formally correct as there is no way information can propagate; but in the Orr-Sommerfeld method the Reynolds number is retained as a finite parameter. In this paper we do not wish to discuss the legitimacy of the Orr-Sommerfeld approach, it is just used as a first approximation to the solution of the linear stability problem at finite Reynolds number. For each time t_0 beyond a certain point (this corresponds to Reynolds number R_{e0}), this problem may be solved to obtain a real wavenumber and phase speed such that the wave is neutral. Above this critical Reynolds number there are two neutral values, and as the Reynolds number increases still further the behaviour of the wave having the smaller wavelength is governed by a five deck structure, the critical layer becomes distinct from the wall layer, due to the size of the phase speed. Our interest is confined to the lower branch where the structure may be described using three decks. An ample description of this structure can be found in Stewartson (1974). The bulk of the boundary layer is referred to as the main deck, which at leading order is inviscid and linear. In the proximity of the wall there is a fully viscous layer, this region contains the critical layer of classical Orr-Sommerfeld theory. Finally there

is an upper deck which is external to the conventional boundary layer and enables matching with the outer potential flow solution. A multiple scales technique is used to resolve the growth of the boundary layer with time.

The temporal growth of the layer produces corrections to the eigenproblems for the wavenumbers and phase speeds at a higher order. The triple deck solutions may be compared with the Orr–Sommerfeld calculation in the regime of common validity, that is where $R_{e0} \gg 1$. In fact the corrections caused by the growth of the layer to the ‘steady’ triple deck scenario occur in the third order equations, one order earlier than those in Smith (1979 a). This is due to the different temporal and spatial scales over which Tollmien–Schlichting waves develop.

The results of this calculation may be viewed in another sense. For a perturbation wavenumber α , at each time t_0 we have a frequency Ω . When the imaginary part of Ω becomes zero, the wave is instantaneously neutral. In the neighbourhood of this neutral time we may develop a weakly nonlinear theory. A Stuart–Watson amplitude equation can be derived, and using this we may determine whether the perturbations are sub or super-critical, that is whether the nonlinearity has a destabilising or stabilising effect on the situation. We shall also consider the interplay between the unsteady and nonlinear effects, as was discussed for the Blasius problem by Smith (1979 b) and Hall & Smith (1984).

The procedure adopted in the remainder of this paper is as follows, in section 2 the basic flow is derived and the quasi-steady problem is formulated. In section 3 the linear triple deck problem is formulated and solved. In section 4 the weakly nonlinear problem is derived and the nonlinear amplitude equation is given. Section 5 contains details of the numerical solution of the linear problem. Section 6 includes discussion of the effect of considering certain small but finite disturbances, and in section 7 some brief conclusions are drawn.

§2 Derivation of the basic flow and the formulation of the quasi-steady problem

The problem is non-dimensionalised in the usual way and the unsteady three dimensional Navier–Stokes equations in cartesian coordinates become (with the temporal scaling $t \sim L/U_\infty$ for the Orr–Sommerfeld modes, where L is a typical lengthscale, and $t \sim L^2/\nu$ for the basic state),

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{R_e} \nabla^2 u,$$

$$\begin{aligned}\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + \frac{1}{R_e} \nabla^2 v, \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{1}{R_e} \nabla^2 w, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,\end{aligned}$$

where R_e is the Reynolds number defined as $U_\infty L/\nu$, and ∇^2 is the Laplacian operator in cartesian coordinates. The choice of the lengthscale L is discussed in section 5, but it shall be taken to be the displacement thickness of the layer, as in Jordinson (1970). In this section disturbances shall be considered to be two dimensional to facilitate comparison between this work and that of Smith (1979 a), to this end it is assumed that $\partial/\partial z \equiv 0$ and $w \equiv 0$.

We require that the unsteady terms and viscous terms balance in the viscous layer, this implies that this layer is of thickness $(\nu t)^{\frac{1}{2}}$. In the non-dimensional coordinates this corresponds to $t^{\frac{1}{2}}$. Thus we introduce a boundary layer similarity variable η given by

$$\eta = \frac{y}{2t^{\frac{1}{2}}}. \quad (2.1)$$

There is no flow in the normal direction, as can be seen from the equation of continuity at leading order. Hence the basic state is governed by

$$\frac{\partial^2 u_B}{\partial \eta^2} + 2\eta \frac{\partial u_B}{\partial \eta} = 0,$$

along with the boundary conditions

$$u_B = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad u_B \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty.$$

The solution of his system is given by

$$u_B = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-q^2} dq,$$

this is an error function.

Now in the context of an Orr-Sommerfeld equation we consider the instant stability of this profile, hence (2.1) now becomes $\eta = y/2t_0^{\frac{1}{2}}$, where t_0 is a constant. We suppose that any perturbation to the flow is independent of the order one temporal scale.

The perturbations are considered to be travelling waves which evolve on the scale L , thus the disturbance is proportional to $\exp(i\theta(x - ct))$. The disturbance is taken to be infinitesimal so that the resulting analysis is linear. As the perturbation is two dimensional, a streamfunction formulation can be exploited, such that

$$U = \frac{\partial\psi}{\partial y}, \quad V = -\frac{\partial\psi}{\partial x}.$$

It should be noted that terms proportional $\partial\psi/\partial t = 0$ have been ignored, so that the equation may be solved as an ordinary differential equation. The governing equation is the conventional Orr-Sommerfeld equation namely,

$$(u_B - c) \left(\frac{\partial^2}{\partial y^2} - \theta^2 \right) \psi - \frac{\partial^2 u_B}{\partial y^2} \psi = \frac{1}{i\theta R_e} \left(\frac{\partial^4}{\partial y^4} - 2\theta^2 \frac{\partial}{\partial y^2} + \theta^4 \right) \psi. \quad (2.2)$$

The boundary conditions imposed are those of no-slip at the wall, and the decay of the disturbance as $y \rightarrow \infty$, this implies the wave is confined to the boundary layer. This eigenvalue problem can be solved numerically employing the method discussed in Malik, Chuang, & Hussaini (1982). The method uses a two-point fourth order compact finite difference scheme based on a Euler-Maclaurin method. A stretched grid was used so that resolution could be retained without using a prohibitive number of points. The calculations were checked by finding the eigenvalues of the adjoint system to (2.2). The critical values were found to be

$$R_{ec} = 2968.4, \quad \Omega_c = 0.13340, \quad \theta_c = 0.4255, \quad c = 0.3135,$$

where $\Omega = \theta c$. This point represents the smallest value of the boundary layer Reynolds number at which the flow situation is linearly unstable to travelling waves of the type considered herein. Plots of θ and c against R_e are given in figures 1 and 2 respectively. These calculation were affected for $t_0 = 1/4$, so that $y = \eta$ at that instance.

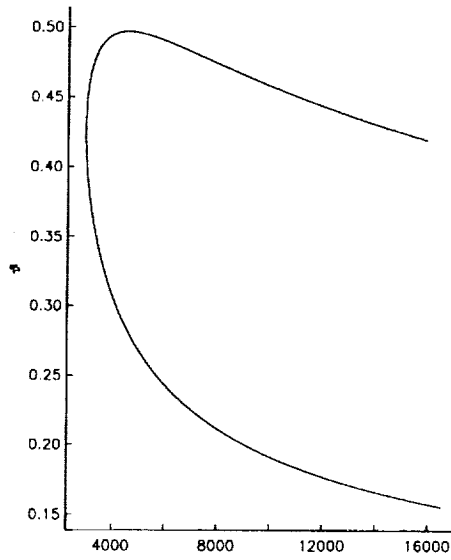


Figure 1: Streamwise wavenumber versus Re , Orr-Sommerfeld calculation

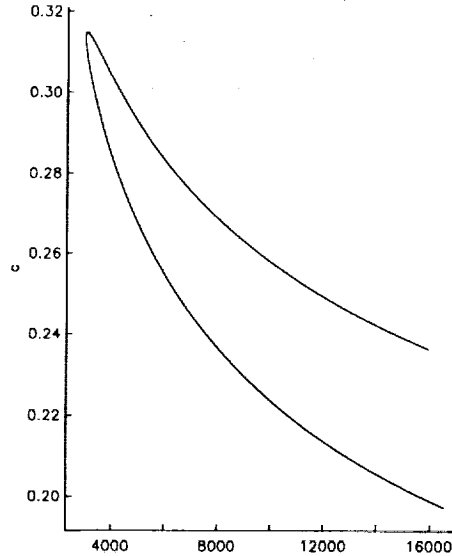


Figure 2: Phase speed versus Re , Orr-Sommerfeld calculation

§3 Formulation and solution of the triple deck problem in the linear regime

In this section the structure of the the linear disturbances is described using triple deck theory. As mentioned previously Re is assumed to be large; and $\epsilon = Re^{-\frac{1}{8}} \ll 1$ is introduced. In classical Orr-Sommerfeld theory the typical wavelength of the neutrally stable modes increases proportionally to $Re^{\frac{1}{8}}$ as $Re \gg 1$, see Stuart (1963). Stuart confirms this asymptotic property with a Reynolds number based on boundary layer thickness. Three distinct regions are considered, namely the upper, main and lower

decks, scaled normally on ϵ^3 , ϵ^4 , and ϵ^5 respectively. Further temporal and spatial variables are introduced, that is T and X , given by, $T = \epsilon^{-2}t$ and $X = \epsilon^{-3}x$. The solutions are considered in the three decks in terms of expansions involving powers of ϵ and $\ln \epsilon$, as in Stewartson (1974). The disturbance is now considered to be proportional to

$$E = \exp \left(i\theta X - i \int_{T_0}^T \Omega(q) dq \right).$$

The disturbance is taken to have slowly varying frequency and constant wavenumber, so that θ and Ω are given by

$$\theta = \theta_1 + \epsilon\theta_2 + \epsilon^2 \ln \epsilon \theta_{3L} + \epsilon^2 \theta_3 + \dots, \quad (3.1a)$$

and

$$\Omega(t) = \Omega_1(t) + \epsilon\Omega_2(t) + \epsilon^2 \ln \epsilon \Omega_{3L}(t) + \epsilon^2 \Omega_3(t) + \dots. \quad (3.1b)$$

A multiple scales form for the temporal derivative is introduced,

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \epsilon^{-2} \frac{\partial}{\partial T}. \quad (3.2)$$

Notice, that to determine the effects of unsteadiness the expansions need only go as far as the fourth order terms not the fifth order terms as in Smith (1979 a), this occurs due to the difference in scalings associated with the temporal and spatial variables.

§3.1 The Main Deck

The transverse coordinate in this layer is $Y = O(1)$, where $Y = \epsilon^{-4}y$. The main deck corresponds to the conventional $R_e^{-\frac{1}{2}}$ boundary layer. The basic flow in the layer is given by the error function as shown in section 2. It should be noted that there is no normal component of this velocity, as can be seen by considering continuity in this layer in the absence of any disturbance. It may be shown that the basic flow has the properties that,

$$u_B(t, Y) \rightarrow \lambda(t)Y + \lambda_3(t)Y^3 + O(Y^5) \quad \text{as} \quad Y \rightarrow 0, \quad (3.3a)$$

$$u_B(t, Y) \rightarrow 1 \quad \text{as} \quad Y \rightarrow \infty, \quad (3.3b)$$

where it is known that $\lambda(t) = \tilde{\lambda}t^{-\frac{1}{2}}$, with $\tilde{\lambda} = 1/\sqrt{\pi}$. The perturbation to the basic state in this layer takes the form,

$$\begin{aligned} u &= [u_1 + \epsilon u_2 + \epsilon^2 \ln \epsilon u_{3L} + \epsilon^2 u_3 + \dots] E, \\ v &= [\epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 \ln \epsilon v_{3L} + \epsilon^3 v_3 + \dots] E, \\ p &= [\epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 \ln \epsilon p_{3L} + \epsilon^3 p_3 + \dots] E. \end{aligned}$$

Notice that the logarithmic terms (characterised by subscript L), occur at one order higher than in the corresponding expansions of Smith (1979 a). This is due to the form of u_B as $Y \rightarrow 0$ given by (3.3 a). This form of the disturbance is now substituted into the governing equations, and nonlinear terms are neglected. The operators M_1 and M_2 are introduced, and are given by

$$M_1(u, v) = i\theta_1 u + \frac{\partial v}{\partial Y}, \quad M_2(u, v) = i\theta_1 u_B u + v \frac{\partial u_B}{\partial Y},$$

where these are identical to those used in Smith (1979 a).

The equations in this layer are conceptually identical to those of Smith (1979 a), except the order at which the logarithmic terms occur. In this deck there is no effect from the multiple scales form to the order considered, so it is sufficient just to quote the equations governing the logarithmic terms,

$$M_1(u_{3L}, v_{3L}) + i\theta_{3L} u_1 = 0, \quad M_2(u_{3L}, v_{3L}) + u_B i\theta_{3L} u_1 = 0, \quad 0 = -\frac{\partial p_{3L}}{\partial Y}.$$

The boundary conditions for the disturbances in this layer are given by matching with the lower and upper decks, the solutions for this layer are considered in section 3.4.

§3.2 The Lower Deck

In this deck the order one variable \tilde{y} is introduced, defined by $\tilde{y} = \epsilon^{-5} y$. The lower deck contains the wall layer and critical layer of classical Orr-Sommerfeld theory. The wave in this deck expands as

$$\begin{aligned} u &= [U_1 + \epsilon U_2 + \epsilon^2 \ln \epsilon U_{3L} + \epsilon^2 U_3 + \dots] E, \\ v &= [\epsilon^2 V_1 + \epsilon^3 V_2 + \epsilon^4 \ln \epsilon V_{3L} + \epsilon^4 V_3 + \dots] E, \\ p &= [\epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 \ln \epsilon P_{3L} + \epsilon^3 P_3 + \dots] E. \end{aligned}$$

The basic flow is given by $u_B \sim \epsilon \tilde{y} \lambda(t) + \epsilon^3 \tilde{y}^3 \lambda_3(t) + \dots$ as can be seen from the form of (3.3 a). From the normal momentum equation it may be shown that the

P_j 's ($j = 1, 2, 3$) are independent of \tilde{y} , but U_j and V_j are functions of \tilde{y} and t . Again following Smith (1979 a) the operators L_1 and L_2 are introduced, and are given by

$$L_1(u, v) = i\theta_1 u + \frac{\partial v}{\partial \tilde{y}}, \quad L_2(u, v, p) = i(\lambda \tilde{y} \theta_1 - \Omega_1)u + \lambda v + i\theta_1 p - \frac{\partial^2 u}{\partial \tilde{y}^2}.$$

Again it is sufficient only to quote those equations that differ from those of Smith (1979 a),

$$\begin{aligned} L_1(U_{3L}, V_{3L}) + i\theta_{3L}U_1 &= 0, \\ L_2(U_{3L}, V_{3L}, P_{3L}) - i\Omega_{3L}U_1 + \lambda \tilde{y} i\theta_{3L}U_1 + i\theta_{3L}P_1 &= 0, \\ L_2(U_3, V_3, P_3) - i(\Omega_2 U_2 + \Omega_3 U_1) + \lambda \tilde{y} i(\theta_2 U_2 + \theta_3 U_1) + i(\theta_2 P_2 + \theta_3 P_1) \\ &\quad + i\lambda_3 \theta_1 U_1 \tilde{y}^3 + 3\tilde{y}^2 \lambda_3 V_1 + \frac{\partial U_1}{\partial t} = 0. \end{aligned}$$

The boundary conditions in this layer are those of no-slip at the wall and matching with the main deck.

§3.3 The Upper deck

In this third deck a coordinate \bar{y} is introduced, defined by $y = \epsilon^3 \bar{y}$, where \bar{y} is an order one quantity, and the disturbance takes the form

$$\begin{aligned} u &= [\epsilon \bar{u}_1 + \epsilon^2 \bar{u}_2 + \epsilon^3 \ln \epsilon \bar{u}_{3L} + \epsilon^3 \bar{u}_3 + \dots] E, \\ v &= [\epsilon \bar{v}_1 + \epsilon^2 \bar{v}_2 + \epsilon^3 \ln \epsilon \bar{v}_{3L} + \epsilon^3 \bar{v}_3 + \dots] E, \\ p &= [\epsilon \bar{p}_1 + \epsilon^2 \bar{p}_2 + \epsilon^3 \ln \epsilon \bar{p}_{3L} + \epsilon^3 \bar{p}_3 + \dots] E. \end{aligned}$$

Here the basic velocity is almost the uniform flow solution that is

$$u_B = 1 + O(\epsilon^3).$$

In this layer all the flow quantities are functions of the slow time scale t and \bar{y} . The operator D is introduced, defined by

$$D(p) = \frac{\partial^2 p}{\partial \bar{y}^2} - \theta_1^2 p.$$

After some algebraic manipulation to remove the velocity components the governing equations for the pressure are obtained as,

$$\begin{aligned} D(\bar{p}_1) &= 0, \quad D(\bar{p}_2) = 2\theta_1 \theta_2 \bar{p}_1, \\ D(\bar{p}_{3L}) &= 2\theta_1 \theta_{3L} \bar{p}_1, \quad D(\bar{p}_3) = 2\theta_1 (\theta_3 \bar{p}_1 + \theta_2 \bar{p}_2) + \theta_2^2 \bar{p}_1. \end{aligned}$$

The boundary conditions are those of matching with the main deck as $\bar{y} \rightarrow 0$ and the requirement that the solution remains bounded as $\bar{y} \rightarrow \infty$. Having set up these equations they are now solved in each layer and the eigen problems are derived. It is worth stressing at this point that there is no effect in the upper deck equations from the unsteadiness (to the order considered). The unsteady derivatives are of a higher order unlike the spatial derivative which occur in the diffusive terms, from

$$\left(\frac{\partial}{\partial x} + \epsilon^{-3} \frac{\partial}{\partial X} \right)^2 \phi.$$

§3.4 Solutions of the main deck equations

The solutions of the first order equations in the main deck can be shown to be

$$v_1 = -i\theta_1 A_1 u_B, \quad u_1 = A_1 \frac{\partial u_B}{\partial Y}, \quad p_1 = P_1,$$

where $P_1 = P_1(t)$ and $A_1 = A_1(t)$ are unknown slowly varying amplitude functions. The pressure in this deck will be matched with the pressure in the lower deck. Solutions here are the same as those given in Smith (1979 a). The integrals H_i defined in Smith (1979 a) are functions of x and Y , whereas here they are functions of t and Y , also obviously u_B is the error function, rather than the solution to the Blasius equation. The logarithmic solutions are given by,

$$u_{3L} = \left(A_{3L} - \frac{A_1 \theta_{3L}}{\theta_1} \right) \frac{\partial u_B}{\partial Y},$$

$$v_{3L} = -i\theta_1 A_{3L} u_B, \quad p_{3L} = P_{3L},$$

§3.5 Solutions in the lower deck

To determine the behaviour of U_1 in this deck, we differentiate the leading order equation of continuity with respect to the decks transverse coordinate, that is \tilde{y} , and eliminate V_1 . It is required that the disturbance is bounded as $\tilde{y} \rightarrow \infty$, thus the solution for $U_{1\xi}$ is found to be given by

$$U_{1\xi} = B_1 \text{Ai}(\xi),$$

where

$$\xi = \Delta^{\frac{1}{3}} \left(\tilde{y} - \frac{\Omega_1}{\lambda \theta_1} \right), \quad \Delta = i\lambda \theta_1,$$

$\text{Ai}(\xi)$ is the Airy function and B_1 is an unknown function of t . Applying the no-slip condition at $\tilde{y} = 0$ in the leading order equation, it is found that

$$B_1 \text{Ai}'_0 \Delta^{\frac{2}{3}} = i\theta_1 P_1, \quad (3.4)$$

where subscript 0 corresponds to evaluation at $\xi = \xi_0 = -i\Omega_1 \Delta^{-\frac{2}{3}}$, and a prime denotes differentiation with respect to ξ . The matching condition on the leading order disturbance as $\tilde{y} \rightarrow \infty$ (therefore as $|\xi| \rightarrow \infty$), is given by $U_1 \rightarrow \lambda A_1$, thus it is found that

$$B_1 \kappa = \lambda A_1,$$

where

$$\kappa = \int_{\xi_0}^{\infty} \text{Ai}(\tilde{\xi}) d\tilde{\xi}.$$

This yields the first relation between the pressure and the negative displacement, P_1 and A_1 respectively, obtained by eliminating B_1 from the equation (3.4). Note that these are effectively the classical critical layer equations, since, as noted previously on the lower branch of the stability curve the critical layer coincides with the wall layer. As can be seen from Smith (1979 a) similar techniques may be used to solve for the higher order equations. The same solutions as Smith (1979 a) are obtained except the unknown amplitudes are now functions of t , rather than x . The equation to be solved at the next order is

$$\frac{\partial^3 U_2}{\partial \xi^3} - \xi \frac{\partial U_2}{\partial \xi} = -i\Omega_2 \Delta^{-\frac{2}{3}} \frac{\partial U_1}{\partial \xi} + i\lambda \left(\xi \Delta^{-\frac{1}{3}} + \frac{\Omega_1}{\lambda \theta_1} \right) \Delta^{-\frac{2}{3}} \theta_2 \frac{\partial U_1}{\partial \xi},$$

which yields the solution

$$U_{2\xi} = B_2 \text{Ai} + \frac{1}{3} \theta_2 \theta_1^{-1} B_1 \text{Ai}''' + i\alpha_{11} B_1 \text{Ai}' \Delta^{-\frac{2}{3}}.$$

The function U_{3L} satisfies

$$\frac{\partial^3 U_{3L}}{\partial \xi^3} - \xi \frac{\partial U_{3L}}{\partial \xi} = -i\Omega_{3L} \Delta^{-\frac{2}{3}} \frac{\partial U_1}{\partial \xi} + i\lambda \left(\xi \Delta^{-\frac{1}{3}} + \frac{\Omega_1}{\lambda \theta_1} \right) \Delta^{-\frac{2}{3}} \theta_{3L} \frac{\partial U_1}{\partial \xi},$$

which gives the solution,

$$U_{3L\xi} = B_{3L} \text{Ai} + \frac{1}{3} \theta_{3L} \theta_1^{-1} B_1 \text{Ai}''' + i\Delta^{-\frac{2}{3}} \left(\frac{\theta_{3L} \Omega_1}{\theta_1} - \Omega_{3L} \right) B_1 \text{Ai}' \Delta^{-\frac{2}{3}}.$$

At next order the function U_3 satisfies

$$\begin{aligned}
\frac{\partial^3 U_3}{\partial \xi^3} - \xi \frac{\partial U_3}{\partial \xi} = & -i\Omega_2 \Delta^{-\frac{2}{3}} \frac{\partial U_2}{\partial \xi} - i\Omega_3 \Delta^{-\frac{2}{3}} \frac{\partial U_1}{\partial \xi} \\
& + \Delta^{-\frac{2}{3}} \left(\xi \Delta^{-\frac{1}{3}} + \frac{\Omega_1}{\lambda \theta_1} \right) \left(\frac{\theta_2}{\theta_1} \frac{\partial U_2}{\partial \xi} + \frac{\theta_3}{\theta_1} \frac{\partial U_1}{\partial \xi} \right) \\
& + \Delta^{-\frac{2}{3}} \frac{\partial}{\partial \xi} \left(\left(\frac{\partial}{\partial t} + \left(\frac{\lambda_t \xi}{\lambda} - \frac{\Delta^{\frac{1}{3}} \Omega_1}{\theta_1 \lambda} \left(\frac{\Omega_{1t}}{\Omega_1} - \frac{\lambda_t}{\lambda} \right) \frac{\partial}{\partial \xi} \right) \right) U_1 \right) \\
& + \frac{6\lambda_3}{i\lambda\theta_1} \Delta^{-\frac{1}{3}} \xi V_1 + 6 \frac{\lambda_3 \Omega_1}{i(\lambda\theta_1)^2} V_1 \\
& + \sum_{j=0}^3 \xi^j a_j \frac{\partial U_1}{\partial \xi},
\end{aligned}$$

which yields the solution

$$\frac{\partial U_3}{\partial \xi} = \sum_{i=0}^6 \beta_i \text{Ai}^{(i)} + B_1 f(\xi),$$

where,

$$\begin{aligned}
f(\xi) = & a_0 \frac{\partial \text{Ai}}{\partial \xi} + \frac{a_1}{3} \frac{\partial^3 \text{Ai}}{\partial \xi^3} + a_2 \left(\frac{1}{7} \frac{\partial^7 \text{Ai}}{\partial \xi^7} - \frac{3}{2} \frac{\partial^4 \text{Ai}}{\partial \xi^4} + 2 \frac{\partial \text{Ai}}{\partial \xi} \right) \\
& + \frac{6\lambda_3}{i\lambda\theta_1} \Delta^{-\frac{1}{3}} \left(\frac{\partial \text{Ai}}{\partial \xi} - \int \int \text{Ai} d\xi d\xi \right) - \frac{6\lambda_3 \Omega_1}{i(\lambda\theta_1)^2} \int \text{Ai} d\xi,
\end{aligned}$$

where the limits of the integrations are ξ to ∞ . In this solution the terms included in the β summation include those of Smith (1979 a) and the unsteady terms. The terms included in the function $f(\xi)$ are those associated with λ_3 , that is the second order mean flow quantity.

The β 's are given by,

$$\begin{aligned}
\beta_0 = B_3, \quad \beta_1 = & \alpha_{11} B_2 + \alpha_{12} B_1 - \frac{\theta_2}{\theta_1} B_1 \alpha_{11} + \Delta^{-\frac{2}{3}} \frac{\lambda_t}{\lambda} B_1, \\
\beta_2 = & \frac{\Delta^{-\frac{2}{3}}}{2} \frac{\partial B_1}{\partial t} + \alpha_{11}^2 B_1 - \frac{\lambda_t}{\lambda} B_1, \quad \beta_3 = \frac{1}{3} \left(\gamma B_1 + \frac{1}{\theta_1} (\theta_2 B_2 + \theta_3 B_1) - \frac{3\theta_2 \alpha_2}{\theta_1} \right), \\
\beta_4 = & \frac{1}{4} \left(\frac{\theta_2}{\theta_1} B_1 \alpha_{11} + \alpha_{11} \alpha_2 \right), \quad \beta_5 = -\frac{\lambda_t}{15\lambda} B_1 \quad \beta_6 = \frac{\theta_2 \alpha_2}{6\theta_1},
\end{aligned}$$

where γ has been introduced, and is given by

$$\gamma = -\Delta^{\frac{1}{3}} \frac{\Omega_1}{\theta_1 \lambda} \left(\frac{\Omega_{1t}}{\Omega_1} - \frac{\lambda_t}{\lambda} \right).$$

The terms a_0 , a_1 , a_2 and a_3 are given by

$$a_0 = \frac{\lambda_3}{\lambda} \Delta^{-\frac{2}{3}}, \quad a_1 = \frac{\lambda_3}{\lambda} \Delta^{-\frac{1}{3}} \frac{3\Omega_1}{\lambda\theta_1}, \quad a_2 = \frac{3\lambda_3}{\lambda} \left(\frac{\Omega_1}{\lambda\theta_1} \right)^2, \quad a_3 = \frac{\lambda_3}{\lambda} \Delta^{\frac{1}{3}} \left(\frac{\Omega_1}{\lambda\theta_1} \right)^3.$$

For brevity α_{1j} and α_2 have been introduced, and are given by

$$\alpha_{1j} = -i\Delta^{-\frac{2}{3}} \left(\frac{\theta_{j+1}\Omega_1}{\theta_1} - \Omega_{j+1} \right) \quad \text{and} \quad \alpha_2 = -\frac{\theta_2}{3\theta_1} B_1(t).$$

Now applying no-slip conditions at the wall it is found that

$$i(\theta_1 P_2 + \theta_2 P_1) = \Delta^{\frac{2}{3}} B_2 \text{Ai}'_0 + \Delta^{\frac{2}{3}} \frac{\theta_2}{3\theta_1} B_1 \text{Ai}''''_0 + \alpha_{11} B_1 \text{Ai}''_0,$$

$$i(\theta_1 P_{3L} + \theta_{3L} P_1) = \Delta^{\frac{2}{3}} B_{3L} \text{Ai}'_0 + \Delta^{\frac{2}{3}} \frac{\theta_{3L}}{3\theta_1} B_1 \text{Ai}''''_0 + i\Delta^{-\frac{2}{3}} \left(\frac{\theta_{3L}\Omega_1}{\theta_1} - \Omega_{3L} \right) B_1 \text{Ai}''_0,$$

$$i(\theta_1 P_3 + \theta_2 P_2 + \theta_3 P_1) = \Delta^{\frac{2}{3}} \sum_{i=0}^6 \beta_i \text{Ai}_0^{(i+1)} + \Delta^{\frac{2}{3}} B_1 \frac{\partial f}{\partial \xi}.$$

Matching with the main deck yields

$$\begin{aligned} B_2 \kappa - \alpha_2 \text{Ai}''_0 - \alpha_{11} B_1 \text{Ai}_0 &= \lambda \left(A_2 - P_1 \hat{I} - \frac{\theta_2}{\theta_1} A_1 \right), \\ B_{3L} \kappa + \frac{\theta_{3L}}{3\theta_1} B_1 \text{Ai}''_0 - i\Delta^{-\frac{2}{3}} \left(\frac{\theta_{3L}\Omega_1}{\theta_1} - \Omega_{3L} \right) B_1 \text{Ai}_0 &= \lambda \left(A_{3L} - P_1 \hat{I} - \frac{\theta_{3L}}{\theta_1} A_1 \right), \\ B_3 \kappa - \Delta^{-\frac{2}{3}} \sum_{i=1}^6 \beta_i \text{Ai}_0^{i-1} - \Delta^{\frac{2}{3}} \int_{\xi_0}^{\infty} f d\xi &= \lambda \left(A_3 - P_2 \hat{I} - \frac{2\Omega_1 P_1}{\theta_1} \tilde{I} - \frac{\theta_2}{\theta_1} A_2 + \frac{\theta_2^2}{\theta_1^2} A_1 - \frac{\theta_3}{\theta_1} A_1 \right). \end{aligned} \quad (3.6)$$

The integrals occurring in the expressions (3.6) are given by

$$\hat{I} = \oint_{\hat{\lambda}} u_B^{-2}(Y_1) dY_1, \quad \text{and} \quad \tilde{I} = \oint_{\hat{\lambda}} u_B^{-3}(Y_1) dY_1,$$

where $\hat{\lambda}$ is arbitrary and non-zero. The horizontal bar denotes that only the (Hadamard) finite part is to be retained. We note that for small η the error function can be approximated by, $\lambda(\eta - \eta^3/3 + \eta^5/5)$, and hence \hat{I} is given by

$$\hat{I} \sim \frac{1}{\lambda^2} \left(\frac{1}{\hat{\lambda}} - \frac{85\hat{\lambda} - 12\hat{\lambda}^3}{26(30 - 10\hat{\lambda}^2 + 3\hat{\lambda}^4)} + \frac{1}{26} \int_{\hat{\lambda}}^0 \frac{435 - 90\eta^2}{30 - 10\eta^2 + 3\eta^4} d\eta \right),$$

with a similar expression for \tilde{I} . In the numerical calculations the quantity $H_{2\infty} - H_{1\infty} - \hat{I}$ is found to be $-0.29512(2t^{\frac{1}{2}})$, this is required in the second order eigenrelation given by (3.8 b).

§3.6 The upper deck solutions and eigenrelations

The pressure equation in the upper deck yields the bounded solution

$$\bar{p}_1 = \bar{P}_1 e^{-\theta_1 \bar{y}},$$

where $\bar{P}_1 = \bar{P}_1(t)$ is an unknown amplitude function. Now matching with the main deck as $\bar{y} \rightarrow 0$ yields

$$\bar{P}_1 = P_1, \quad \theta_1 \bar{P}_1 = \theta_1^2 A_1 (\equiv i\theta_1 v_{1\infty}),$$

where $v_{1\infty}$ denotes the values of v_1 (the main deck normal velocity) as the decks transverse coordinate (Y) tends to infinity. This provides the leading order eigenvalue problem

$$i\theta_1^2 \kappa = \lambda \Delta^{\frac{2}{3}} \text{Ai}'_0. \quad (3.7)$$

This relationship determines the wavenumber in terms of the frequency and the skin friction λ . The higher order pressure functions may be determined, and are given by

$$\begin{aligned} \bar{p}_2 &= (\bar{P}_2 - \theta_2 P_1 \bar{y}) e^{-\theta_1 \bar{y}}, \\ \bar{p}_{3L} &= (\bar{P}_{3L} - \theta_{3L} P_1 \bar{y}) e^{-\theta_1 \bar{y}}, \\ \bar{p}_3 &= \left(\bar{P}_3 - (\theta_3 P_1 + \theta_2 \bar{P}_2) \bar{y} + \frac{1}{2} \theta_2^2 P_1 \bar{y}^2 \right) e^{-\theta_1 \bar{y}}. \end{aligned}$$

To match with the main deck, the unknown functions must satisfy

$$\begin{aligned} \bar{P}_2 &= P_2 - A_1 \theta_1^2 H_{1\infty}, \\ i\theta_1 P_1 H_{2\infty} + 2i\Omega_1 A_1 - i\theta_1 A_2 &= iA_1 \theta_2 - i\bar{P}_2 - i\frac{\theta_2}{\theta_1} P_1, \\ \bar{P}_{3L} &= P_{3L}, \\ -\theta_1 A_{3L} &= i\theta_{3L} A_1 - i\bar{P}_{3L} - \frac{i\theta_{3L} P_1}{\theta_1}, \\ \bar{P}_3 &= P_3 + \theta_1^2 P_1 H_{3\infty} - \theta_1 (\theta_1 A_2 + \theta_2 A_1) H_{1\infty} + 2\Omega_1 \theta_1 A_1 H_{4\infty}, \end{aligned}$$

$$\begin{aligned}
& i(\theta_2 P_1 + \theta_1 P_2) H_{2\infty} - i\theta_1^3 A_1 H_{5\infty} + i\Omega_1 P_1 H_{6\infty} + i\Omega_1 \left(A_2 - \frac{\theta_2 A_1}{\theta_1} \right) \\
& + i\Omega_2 A_1 - i\theta_1 A_3 = \frac{1}{\theta_1} ((\Omega_1 - \theta_2) v_{2\infty} + (\Omega_2 - \theta_3) v_{1\infty}) - i\bar{P}_3 - i\frac{\theta_3 P_1}{\theta_1} - i\frac{\theta_2 \bar{P}_2}{\theta_1}.
\end{aligned}$$

The higher order eigenrelations may now be obtained,

$$\hat{D}\theta_2 - \Omega_2\theta_1 \frac{Ai_0 D}{\kappa\Delta^{\frac{2}{3}}} = i\theta_1^2 (H_{2\infty} - H_{1\infty} - \hat{I}) + 2i\Omega_1, \quad (3.8a)$$

$$\hat{D}\theta_{3L} - \Omega_{3L}\theta_1 \frac{Ai_0 D}{\kappa\Delta^{\frac{2}{3}}} = 6\frac{\lambda_3 i\Omega_1}{\lambda^4}, \quad (3.8b)$$

$$\hat{D}\theta_3 - \Omega_3\theta_1 \frac{Ai_0 D}{\kappa\Delta^{\frac{2}{3}}} = i(c.t.s + c.t.\lambda_3 + n.t.), \quad (3.8c)$$

where D and \hat{D} are given by,

$$D = 1 + \frac{\kappa\xi_0}{Ai_0'}, \quad \text{and} \quad \hat{D} = \frac{4}{3}i + \frac{2}{3} \left(\frac{\Omega_1 Ai_0}{\kappa\Delta^{\frac{2}{3}}} \right).$$

In (3.8) *c.t.s* denotes the terms that arise in conventional analysis; these are exactly the same as those occurring in the third order terms of Smith (1979 a), as the non-parallelism has no effect at this order. These terms can be directly determined from (3.15 b) of Smith (1979 a). The terms *c.t. λ_3* are those associated with the second order basic flow terms and *n.t.* are the terms that arise due to the *t*-derivatives (that is unsteadiness of the modes). Discussion of the solution of these eigenrelations is included in section 5.

§4 Nonlinear theory for larger disturbances

We start by setting up the nonlinear problem for an unsteady triple deck. As in the linear problem the scalings $x = \epsilon^3 X$ and $t = \epsilon^2 T$ are introduced. In the Rayleigh layer under consideration at the time station t , the flow field in the absence of any disturbance is given by

$$u = u_B(t, Y) + \dots,$$

where $u_B(t, Y)$ is given by the error function. It is also worth re-iterating at this point that as $Y \rightarrow 0$,

$$u_B(t, Y) \rightarrow \lambda(t)Y + \lambda_3(t)Y^3 + \dots,$$

where $\lambda(t) = (\pi t)^{-\frac{1}{2}}$.

The properties of the entire problem may now be determined by examining the lower deck, the same transverse coordinate is used as in section 3.2, that is $y = \epsilon^5 \tilde{y}$, where $\tilde{y} = O(1)$. In the lower deck the flow is taken to be of the form,

$$\mathbf{u} = (\epsilon U(X, \tilde{y}, Z, T; t), \epsilon^3 V(X, \tilde{y}, Z, T; t), \epsilon W(X, \tilde{y}, Z, T; t), \epsilon^2 P(X, Z, T; t)).$$

This flow includes both the basic flow and any disturbance quantity. The fact that the pressure P is independent of \tilde{y} is given by the wall normal momentum equation. The governing equations in the deck are now found to be

$$\frac{\partial U}{\partial T} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial \tilde{y}} + W \frac{\partial U}{\partial Z} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial \tilde{y}^2}, \quad (4.1a)$$

$$\frac{\partial W}{\partial T} + U \frac{\partial W}{\partial X} + V \frac{\partial W}{\partial \tilde{y}} + W \frac{\partial W}{\partial Z} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial \tilde{y}^2}, \quad (4.1b)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial \tilde{y}} + \frac{\partial W}{\partial Z} = 0, \quad (4.1c)$$

with boundary conditions

$$U = V = W = 0 \quad \text{at} \quad \tilde{y} = 0, \quad (4.1d)$$

and,

$$U \sim \lambda(\tilde{y} + A(X, Z, T)) \quad \text{as} \quad \tilde{y} \rightarrow \infty. \quad (4.1e)$$

Here it is assumed that the spanwise component of the disturbance is confined to the lower deck. From the upper deck equations we see that

$$\nabla^2 \bar{p} = 0, \quad \text{with} \quad \bar{p} = P(X, Z, T) \quad \text{and} \quad \frac{\partial \bar{p}}{\partial \bar{y}} = \frac{\partial^2 A}{\partial X^2} \quad \text{at} \quad \bar{y} = 0$$

where \bar{y} is the transverse upper deck coordinate, and \bar{p} is a function of X , \bar{y} , Z and T , and is bounded as $\bar{y} \rightarrow \infty$. Now consider a small perturbation to the basic state ($U = \lambda \tilde{y}$, $V = W = P = A = 0$) in the lower deck. The perturbation to this system is taken to be of order $h \ll 1$. In Hall & Smith (1984) initially a bi-modal analysis was considered and subsequently a multiple mode analysis, here a single mode is considered. It is assumed that h is larger than any positive power of ϵ so the governing equations can be assumed to be given by (4.1) for all orders of h . In section 6, the problem

involved in making $h \sim \epsilon^n$ is discussed, where unsteady effects occur at the same order as nonlinear effects. The flow is now perturbed and taken to be given by

$$\begin{aligned} U &= \lambda \tilde{y} + hU_1 + h^2U_2 + h^3U_3 + \dots, \\ V &= hV_1 + h^2V_2 + h^3V_3 + \dots, \\ W &= hW_1 + h^2W_2 + h^3W_3 + \dots, \\ P &= hP_1 + h^2P_2 + h^3P_3 + \dots, \\ A &= hA_1 + h^2A_2 + h^3A_3 + \dots, \\ \bar{p} &= h\bar{p}_1 + h^2\bar{p}_2 + h^3\bar{p}_3 + \dots, \end{aligned}$$

The disturbance is considered to be proportional to E and its integer multiples, where E is given by

$$E \equiv \exp(i(\theta X + \gamma Z - \Omega T)),$$

θ , γ and Ω are real constants for neutral stability, and Z represents the spanwise variation on the $O(R_e^{-\frac{3}{8}})$ scale. For any spanwise wavenumber γ a pair θ and Ω may be determined using the first order eigenrelation given by linear theory. At this point a new time coordinate is introduced, namely \tilde{T} defined by

$$T = \frac{\Omega}{\Omega_1} \bar{T} = h^{-2} \tilde{T}.$$

This necessitates the inclusion of further terms in the skin friction in the neighbourhood of the neutral time t_0 , as mentioned in Smith (1980); the correction to Smith (1979 b), so

$$t = t_0 + h^2 \tilde{t}, \quad \lambda = \lambda + h^2 \lambda_2 + O(h^3), \quad \Omega = \Omega_1 + h^2 \Omega_2 + O(h^3),$$

where $\lambda_2 = \tilde{t}(d\lambda/dt)|_{t=t_0}$. It is necessary to include the further terms in the Ω form, to satisfy the third order eigenrelation.

Now substitute the disturbance into the governing equations including the necessary multiple scales approach for temporal derivatives. The equation of continuity at successive powers of h becomes,

$$L_1(U_1, V_1, W_1) = 0, \quad L_1(U_2, V_2, W_2) = 0, \quad L_1(U_3, V_3, W_3) = 0.$$

The streamwise momentum equation at successive orders yields,

$$L_2(U_1, V_1, P_1) = 0,$$

$$\begin{aligned}
L_2(U_2, V_2, P_2) &= - \left(U_1 \frac{\partial U_1}{\partial X} + V_1 \frac{\partial U_1}{\partial Y} + W_1 \frac{\partial U_1}{\partial Z} \right), \\
L_2(U_3, V_3, P_3) &= - \frac{\partial U_1}{\partial \tilde{T}} - \lambda_2 Y \frac{\partial U_1}{\partial X} - V_1 \lambda_2 - U_1 \frac{\partial U_2}{\partial X} - U_2 \frac{\partial U_1}{\partial X} \\
&\quad - V_1 \frac{\partial U_2}{\partial Y} - V_2 \frac{\partial U_1}{\partial Y} - W_1 \frac{\partial U_2}{\partial Z} - W_2 \frac{\partial U_1}{\partial Z}.
\end{aligned}$$

The spanwise momentum equation become,

$$\begin{aligned}
L_3(W_1, P_1) &= 0, \\
L_3(W_2, P_2) &= -U_1 \frac{\partial W_1}{\partial X} - V_1 \frac{\partial W_1}{\partial Y} - W_1 \frac{\partial W_1}{\partial Z}, \\
L_3(W_3, P_3) &= - \frac{\partial W_1}{\partial \tilde{T}} - \lambda_2 Y \frac{\partial W_1}{\partial X} - U_1 \frac{\partial W_2}{\partial X} - U_2 \frac{\partial W_1}{\partial X} \\
&\quad - V_1 \frac{\partial W_2}{\partial Y} - V_2 \frac{\partial W_1}{\partial Y} - W_1 \frac{\partial W_2}{\partial Z} - W_2 \frac{\partial W_1}{\partial Z}.
\end{aligned}$$

The operators L_1 , L_2 and L_3 are defined by,

$$\begin{aligned}
L_1(U, V, W) &= \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z}, \\
L_2(U, V, P) &= \frac{\partial U}{\partial \tilde{T}} + \lambda Y \frac{\partial U}{\partial X} + \lambda V + \frac{\partial P}{\partial X} - \frac{\partial^2 U}{\partial Y^2}, \\
L_3(W, P) &= \frac{\partial W}{\partial \tilde{T}} + \lambda Y \frac{\partial W}{\partial X} + \frac{\partial P}{\partial Z} - \frac{\partial^2 W}{\partial Y^2}.
\end{aligned}$$

The boundary conditions at successive orders are given by

$$\begin{aligned}
U_i = V_i = W_i &= 0 \quad \text{at} \quad \tilde{y} = 0 \quad (i = 1, 2, 3), \\
U_1 &\rightarrow \lambda A_1(X, Z, T) \quad \text{as} \quad \tilde{y} \rightarrow \infty, \\
U_2 &\rightarrow \lambda A_2(X, Z, T) \quad \text{as} \quad \tilde{y} \rightarrow \infty, \\
U_3 &\rightarrow \lambda A_3(X, Z, T) + \lambda_2 A_1(X, Z, T) \quad \text{as} \quad \tilde{y} \rightarrow \infty,
\end{aligned}$$

It is worth noting another significant discrepancy between this work and that considering a spatially growing boundary layer, that is the slow time derivative does not come into the upper deck governing equations, as was noted in the linear section. Now consider U_1 to be wavelike proportional to E and its conjugate denoted by \bar{E} . The first order disturbance field is now given by

$$U_1 = \tilde{U}_1(\tilde{y}, \tilde{T}) E + c.c., \quad V_1 = \tilde{V}_1(\tilde{y}, \tilde{T}) E + c.c., \quad W_1 = \tilde{W}_1(\tilde{y}, \tilde{T}) E + c.c.,$$

$$P_1 = \tilde{P}_1 (\tilde{T}) E + c.c., \quad \bar{p}_1 = \bar{p}_{11} (\tilde{y}, \tilde{T}) E + c.c., \quad A_1 = \tilde{A}_1 (\tilde{T}) E + c.c.,$$

where *c.c.* denotes complex conjugate and the disturbance quantities with tildes are independent of *X*, *Z*, and *T*. By combining the first order equations the two governing equations are obtained

$$\frac{\partial^2 \tilde{W}_1}{\partial \xi^2} - \xi \tilde{W}_1 = \frac{1}{\Delta^{\frac{2}{3}}} \frac{\tilde{P}_1 \gamma}{\theta \lambda}, \quad \frac{\partial^3 \tilde{U}_1}{\partial \xi^3} - \xi \frac{\partial \tilde{U}_1}{\partial \xi} = -\frac{\lambda \gamma i}{\Delta^{\frac{2}{3}}} \tilde{W}_1, \quad (4.2a, b)$$

where

$$\xi = \Delta^{\frac{1}{3}} \left(\tilde{y} - \frac{\Omega}{\theta \lambda} \right), \quad \Delta = i \lambda \theta. \quad (4.2c)$$

At this point it is advantageous to introduce a further disturbance quantity that is χ_{ij} . The equations (4.2 a) and (4.2 b) may be manipulated and it is found that χ_{11} satisfies

$$\frac{\partial^3 \chi_{11}}{\partial \xi^3} - \xi \frac{\partial \chi_{11}}{\partial \xi} = 0, \quad (4.3a)$$

where

$$\chi_{11} = i \theta U_1 + i \gamma W_1. \quad (4.3b)$$

The solution of (4.3 a) together with a bounded condition at infinity yields,

$$\chi_{11} = B_1 (\tilde{T}) \text{Ai}(\xi). \quad (4.3c)$$

The three dimensional eigenrelation may now be obtained, namely

$$\lambda^2 \text{Ai}'(\xi_0) = (i \theta \lambda)^{\frac{1}{3}} (\theta^2 + \gamma^2)^{\frac{1}{2}} \kappa, \quad (4.4a)$$

as given in Hall & Smith (1984), where

$$\xi_0 = -\frac{\Omega_1 i^{\frac{1}{3}}}{(\theta \lambda)^{\frac{2}{3}}} \quad \text{and} \quad \kappa = \int_{\xi_0}^{\infty} \text{Ai}(q) dq. \quad (4.4b, c)$$

The neutral solution of the eigenvalue problem is given by

$$\xi_0 = -2.298 i^{\frac{1}{3}}.$$

In this relationship θ is real and (4.4 a) yields

$$\theta^{\frac{1}{3}} (\theta^2 + \gamma^2)^{\frac{1}{2}} = 1.001 \lambda^{\frac{5}{3}},$$

and,

$$\Omega_1 = 2.299 \theta^{\frac{2}{3}} \lambda^{\frac{5}{3}}.$$

Thus for a particular spanwise wavenumber γ , a pair θ and Ω_1 maybe obtained such that neutrality is assured, a plot of the variation with γ of θ and Ω_1 is given in figure 3.

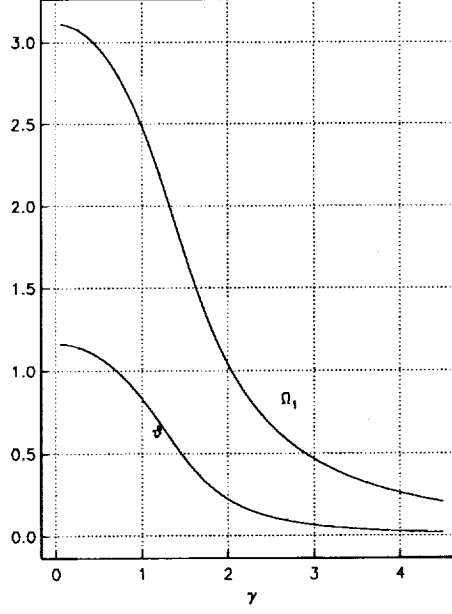


Figure 3: Variation of θ and Ω_1 with γ .

The solution of the second order problem is now considered, the disturbance quantities now contain second harmonic terms, the disturbance is taken to be of the form

$$U_2 = E\tilde{U}_{21} + E^{-1}\tilde{U}_{21}^{(c)} + E^2\tilde{U}_{22} + E^{-2}\tilde{U}_{22}^{(c)} + \tilde{U}_{20},$$

with similar expansions for V_2 , W_2 , P_2 , \bar{p}_2 and A_2 . The terms proportional to E^2 are considered and it is found that $(\tilde{U}_{22}, \tilde{V}_{22}, \tilde{W}_{22}, \tilde{P}_{22})$ satisfies the system

$$2i\theta\tilde{U}_{22} + \frac{\partial\tilde{V}_{22}}{\partial\tilde{y}} + 2i\gamma\tilde{W}_{22} = 0,$$

$$-2i\Omega\tilde{U}_{22} + \lambda\tilde{y}2i\theta\tilde{U}_{22} + \lambda\tilde{V}_{22} + 2i\theta\tilde{P}_{22} - \frac{\partial^2\tilde{U}_{22}}{\partial\tilde{y}^2} = -i\theta\tilde{U}_1^2 - V_1\frac{\partial\tilde{U}_1}{\partial\tilde{y}} - i\gamma\tilde{W}_1\tilde{U}_1,$$

$$-2i\Omega\tilde{W}_{22} + \lambda\tilde{y}2i\theta\tilde{W}_{22} + 2i\gamma\tilde{P}_{22} - \frac{\partial^2\tilde{W}_{22}}{\partial\tilde{y}^2} = -i\theta\tilde{U}_1\tilde{W}_1 - V_1\frac{\partial\tilde{U}_1}{\partial\tilde{y}} - i\gamma\tilde{W}_1^2,$$

along with boundary conditions

$$\tilde{U}_{22} = \tilde{V}_{22} = \tilde{W}_{22} = 0 \quad \text{at} \quad \tilde{y} = 0,$$

$$\tilde{U}_{22} \rightarrow \lambda\tilde{A}_{22} \quad \text{as} \quad \tilde{y} \rightarrow \infty.$$

Again at this point it is advantageous to re-introduce χ_{ij} , this time χ_{22} defined by

$$\chi_{22} = 2i\theta\tilde{U}_{22} + 2i\gamma\tilde{W}_{22},$$

so the system becomes,

$$\frac{\partial^2 \chi_{22}}{\partial \xi^3} - 2\xi \frac{\partial \chi_{22}}{\partial \xi} = - \left(\chi_{11} \frac{\partial \chi_{11}}{\partial \xi} + V_1 \frac{\partial^2 \chi_{11}}{\partial \xi^2} \right).$$

The solution of which necessitates the introduction of $\hat{\xi}$, given by $\hat{\xi} = 2^{-\frac{1}{3}}\xi$. Note that it is not necessary to solve the equation to determine $(\tilde{U}_{21}, \tilde{V}_{21}, \tilde{W}_{21}, \tilde{P}_{21})$, as it can be seen that the governing equations for these quantities are the homogeneous equations (4.2); the solution is a linear multiple of the first order solution. Now finally at this order the mean adjustment components are determined by

$$\frac{\partial \tilde{V}_{20}}{\partial \tilde{y}} = 0,$$

$$\begin{aligned} \lambda \tilde{V}_{20} - \frac{\partial^2 \tilde{U}_{20}}{\partial \tilde{y}^2} &= -\tilde{V}_1 \frac{\partial \tilde{U}_1^{(c)}}{\partial \tilde{y}} - \tilde{V}_1^{(c)} \frac{\partial \tilde{U}_1}{\partial \tilde{y}} - i\gamma \left(\tilde{W}_1^{(c)} \tilde{U}_1 - \tilde{W}_1 \tilde{U}_1^{(c)} \right), \\ -\frac{\partial^2 \tilde{W}_{20}}{\partial \tilde{y}^2} &= -i\theta \left(\tilde{W}_1^{(c)} \tilde{U}_1 - \tilde{W}_1 \tilde{U}_1^{(c)} \right) - \tilde{V}_1^{(c)} \frac{\partial \tilde{W}_1}{\partial \tilde{y}} - \tilde{V}_1 \frac{\partial \tilde{W}_1^{(c)}}{\partial \tilde{y}}, \end{aligned}$$

with boundary conditions

$$\tilde{U}_{20} = \tilde{V}_{20} = \tilde{W}_{20} = 0 \quad \text{at} \quad \tilde{y} = 0.$$

By the usual substitution, that is $\chi_{20} = i\theta U_{20} + i\gamma W_{20}$, it is shown that

$$\frac{\partial V_{20}}{\partial \xi} = 0,$$

$$\Delta \frac{\partial^3 \chi_{20}}{\partial \xi^3} = \Delta^{\frac{2}{3}} \bar{V}_1 \frac{\partial^2 \chi_{11}}{\partial \xi^2} - \Delta^{\frac{2}{3}} V_1 \frac{\partial^2 \bar{\chi}_{11}}{\partial \xi^2} + \Delta^{\frac{1}{3}} \frac{\partial}{\partial \xi} (\chi_{11} \bar{\chi}_{11}).$$

Finally the third order equations are considered and again by elementary analysis it is known that the third order quantities contain terms proportional to E^3 , E^2 , E^1 , E^0 , \bar{E}^1 , \bar{E}^2 and \bar{E}^3 , however, determination of the functions associated with E is sufficient

to give the required amplitude equation. The functions $(\tilde{U}_{31}, \tilde{V}_{31}, \tilde{W}_{31}, \tilde{P}_{31})$ are given by the solution of the system

$$\begin{aligned} \frac{\partial^3 \tilde{U}_{31}}{\partial \xi^3} - \xi \frac{\partial \tilde{U}_{31}}{\partial \xi} &= \frac{1}{\Delta^{\frac{2}{3}}} \frac{\partial \tilde{U}_1}{\partial \tilde{T}} - \frac{\gamma}{\theta} \tilde{W}_{31} + \frac{\lambda_2 i \theta}{\Delta^{\frac{2}{3}}} \left(\Delta^{-\frac{1}{3}} \xi \frac{\partial \tilde{U}_1}{\partial \xi} + \frac{\Omega_1}{\lambda \theta} \frac{\partial \tilde{U}_1}{\partial \xi} - \Delta^{-\frac{1}{3}} \frac{\gamma}{\theta} \right) \\ &\quad - \frac{1}{\Delta^{\frac{2}{3}}} \frac{\partial}{\partial \xi} \left(i \theta \left(\tilde{U}_{22} \tilde{U}_1^{(c)} + \tilde{U}_{20} \tilde{U}_1 \right) \right. \\ &\quad \left. + \Delta^{\frac{1}{3}} \left(\tilde{V}_1^{(c)} \frac{\partial \tilde{U}_{22}}{\partial \xi} + \tilde{V}_1 \frac{\partial \tilde{V}_{20}}{\partial \xi} + \tilde{V}_{22} \frac{\partial \tilde{U}_1^{(c)}}{\partial \xi} + \tilde{V}_{20} \frac{\partial \tilde{U}_1}{\partial \xi} \right) \right. \\ &\quad \left. + i \gamma \left(2 \tilde{U}_{22} \tilde{W}_1^{(c)} - \tilde{W}_{22} \tilde{U}_1^{(c)} + \tilde{W}_{20} \tilde{W}_1 \right) \right), \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \tilde{W}_{31}}{\partial \xi^2} - \xi \tilde{W}_{31} &= \frac{1}{\Delta^{\frac{2}{3}}} \frac{\partial \tilde{W}_1}{\partial \tilde{T}} + \frac{i \gamma}{\Delta^{\frac{2}{3}}} \tilde{P}_{31} + \frac{\lambda_2 i \theta}{\Delta^{\frac{2}{3}}} \left(\Delta^{-\frac{1}{3}} \xi + \frac{\Omega_1}{\lambda \theta} \right) \tilde{W}_1 \\ &\quad + \frac{i \theta}{\Delta^{\frac{2}{3}}} \left(2 \tilde{W}_{22} \tilde{U}_1^{(c)} - \tilde{U}_{22} \tilde{W}_1^{(c)} + \tilde{U}_{20} \tilde{W}_1 \right) \\ &\quad + \frac{1}{\Delta^{\frac{1}{3}}} \left(\tilde{V}_1^{(c)} \frac{\partial \tilde{W}_{22}}{\partial \xi} + \tilde{V}_1 \frac{\partial \tilde{W}_{20}}{\partial \xi} + \tilde{V}_{22} \frac{\partial \tilde{W}_1^{(c)}}{\partial \xi} + \tilde{V}_{20} \frac{\partial \tilde{W}_1}{\partial \xi} \right) \\ &\quad + \frac{i \gamma}{\Delta^{\frac{2}{3}}} \left(\tilde{W}_{22} \tilde{W}_1^{(c)} + \tilde{W}_{20} \tilde{W}_1 \right), \end{aligned}$$

with boundary conditions

$$\tilde{U}_{31} = \tilde{V}_{31} = \tilde{W}_{31} = 0 \quad \text{at} \quad \tilde{y} = 0,$$

$$\tilde{U}_{31} \rightarrow \lambda A_{31} + \lambda_2 A_{11} \quad \text{as} \quad \tilde{y} \rightarrow \infty.$$

We make the substitution $\chi_{31} = i \theta \tilde{U}_{31} + i \gamma \tilde{W}_{31}$ and it is found that

$$\begin{aligned} \frac{\partial^3 \chi_{31}}{\partial \xi^3} - \xi \frac{\partial \chi_{31}}{\partial \xi} &= \Delta^{-\frac{2}{3}} \frac{\partial^2 \chi_{11}}{\partial \xi \partial \tilde{T}} + \Delta^{\frac{1}{3}} \frac{\lambda_2}{\lambda} \left(\xi \Delta^{-\frac{1}{3}} + \frac{\Omega_1}{\lambda \theta} \right) \frac{\partial \chi_{11}}{\partial \xi} - i \Omega_2 \Delta^{-\frac{2}{3}} \frac{\partial \chi_{11}}{\partial \xi} \\ &\quad + \left(\frac{\Delta^{\frac{1}{3}}}{2} \bar{\chi}_{11} \frac{\partial \chi_{22}}{\partial \xi} - \Delta^{\frac{2}{3}} V_1 \frac{\partial^2 \chi_{20}}{\partial \xi^2} - \frac{\Delta^{\frac{2}{3}}}{2} \bar{V}_1 \frac{\partial^2 \chi_{22}}{\partial \xi^2} + \Delta^{\frac{2}{3}} V_{22} \frac{\partial^2 \bar{\chi}_{11}}{\partial \xi^2} \right. \\ &\quad \left. + \Delta^{\frac{1}{3}} \frac{\partial \bar{\chi}_{11}}{\partial \xi} \chi_{22} - \Delta^{\frac{1}{3}} \frac{\partial \chi_{11}}{\partial \xi} \chi_{20} \right) \end{aligned}$$

We require a solvability condition for this equation (with the upper deck terms), which yields

$$\frac{dB_1}{d\tilde{T}} = \sigma_1 B_1 + \sigma_2 B_1 |B_1|^2, \quad (4.5)$$

where B_1 is the lower deck amplitude function, and where σ_2 is an integral given in the appendix (equation (A.1)). Note that σ_1 may be thought of as a frequency perturbation, and if $\sigma_1 = i\tilde{\Omega}$, we may determine $\tilde{\Omega}$ from the eigenrelation (4.4 a), along with (4.4 b,c). If Ω is replaced $\Omega + h^2\bar{\Omega}$ and λ by $\lambda + h^2\bar{\lambda}$ ($\bar{\Omega} = \Omega_2$ and $\bar{\lambda} = \lambda_2$), we obtain the relationship,

$$\frac{\bar{\Omega}}{\Omega} = \frac{\bar{\lambda}}{\lambda} \frac{\left(\frac{5Ai'\lambda^{\frac{5}{3}}}{\xi_0 Ai} - \frac{2}{3}\xi_0\lambda^{\frac{5}{3}} + \frac{2}{3}(i\theta)^{\frac{1}{3}}(\theta^2 + \gamma^2)^{\frac{1}{2}} \right)}{\left((i\theta)^{\frac{1}{3}}(\theta^2 + \gamma^2)^{\frac{1}{2}} - \lambda^{\frac{5}{3}}\xi_0 \right)}. \quad (4.6)$$

Note that it is not necessary to calculate the value of σ_2 as it has the same value as in Hall & Smith (1984), where it is been assumed that λ is unity. The velocity field has been normalised so that $\frac{dU_1}{dY}|_{Y=0} = 1$. These results are for a single mode, whereas Hall & Smith (1984) considered two modes, note that the value of a_{1r} (which corresponds to σ_{2r} here) is independent of the second spanwise wavenumber thus,

γ	4.2	3.9	3.6	3.3	3.0	2.7	2.4	2.1	1.8	1.5	1.2
σ_{2r}	-0.0039	-0.0049	-0.0062	-0.0081	-0.0107	-0.0147	-0.0214	-0.0314	-0.0515	-0.0919	-0.2250

Table 1, γ against σ_{2r} , as in Hall & Smith (1984).

Note that $\bar{\Omega} = q_1(\tilde{t} - t_n)$ and $\bar{\lambda} = \lambda'(t_0)(\tilde{t} - t_n)$, hence the relationship (4.6) becomes

$$q_{1i} = - \frac{\lambda'(t_0) \left(1.46912\lambda^{\frac{5}{3}} \right)}{\lambda(t_0) \left(2.2972\theta^{\frac{1}{3}}(\theta^2 + \gamma^2)^{\frac{1}{2}} + 5.266128\lambda^{\frac{5}{3}} \right)},$$

where a superscript prime represents differentiation with respect to t . It is noted that $\lambda'(t_0) < 0$, thus $\tilde{\Omega}_i > 0$ after the neutral time $\tilde{t} = t_n$, as is expected from linear theory.

Note that by considering smaller disturbances unsteady effects may be incurred. It is assumed that h is small, but not so small that the leading order effects described in this section are changed. In fact the calculations included in this section are appropriate to $1 \gg h \gg R_e^{-\frac{1}{16}}$. The case where $h \sim R_e^{-\frac{1}{16}}$ is now considered, note that it is similar to the case discussed in Hall & Smith (1984), for the spatial problem. It is

taken that $h = HR_e^{-\frac{1}{16}}$, with H of order one, and corresponding to a disturbance size of $\delta = HR_e^{-\frac{3}{16}}$, and it is found that the nonlinear amplitude equation is given by,

$$\frac{dB_1}{d\tilde{T}} = iq_1 \left(-t_1 + \tilde{t} + \tilde{T}H^{-4} \right) B_1 + \sigma_2 B_1 |B_1|^2. \quad (4.7)$$

The coefficients in this equations remain the same as in (4.5) so no further numerical work is needed, the discussion of the solutions of this equation are included in section 6.

§5 The neutral stability characteristics

§5.1 Quasi steady flow stability in the triple deck (linear case)

Here we find the wave characteristics that can be determined from the eigenrelations (3.7 & 8). It should be stressed at this point that triple deck theory is only applicable for $R_e \gg 1$, however we can compute the neutral wavenumber θ_n and compare it with the Orr-Sommerfeld predictions. The eigenrelations (3.7,8a & b) are unaffected by the boundary layer's growth, so the condition of neutrality is realised by ensuring that Ω_1 , Ω_2 and Ω_{3L} are real.

To ensure that Ω_1 and θ_1 are real we calculate the value of ξ_0 (defined in section 3.5), required to make the quantity $Ai'_0/(i^{\frac{2}{3}}\kappa)(=\hat{c}_1 = 1.00065)$ real. This occurs when $\xi_0 \sim -2.298i^{\frac{1}{3}} = -i^{\frac{1}{3}}\hat{\xi}_0$, which yields

$$\Omega_1 = \lambda_1^{\frac{3}{2}} \hat{\xi}_0 \hat{c}_1^{\frac{1}{2}} = \lambda_1^{\frac{3}{2}} \hat{\Omega}_1, \quad \theta_1 = \lambda_1^{\frac{5}{4}} \hat{c}_1^{\frac{3}{4}} = \lambda_1^{\frac{5}{4}} \hat{\theta}_1.$$

At this point we recall that $\lambda_1 = \frac{1}{\sqrt{\pi t}}$; but θ_1 is a constant so we can think of this expression as defining the neutral time,

$$t_n \sim \frac{\hat{c}_1}{\pi} \theta_1^{-\frac{8}{5}}.$$

After this time modes will exhibit temporal amplification, and the weakly nonlinear analysis described herein will come into effect as the mode attains a finite amplitude. It appears that the higher wavenumber modes will start to grow earliest, of course this can be seen from figure 1. These results may also be interpreted in terms of the local boundary layer Reynolds number, defined by $R_{\delta^*} = 2/\sqrt{\pi}(tR_e)^{\frac{1}{2}}$, where $2\sqrt{t}/\sqrt{\pi}$ is the displacement thickness. As noted in section 2, the Orr-Sommerfeld calculations

are affected for $t = 1/4$ so that the value for the displacement thickness is $1/2\sqrt{\pi}$. We can express the neutrally stable wavenumber in terms of this Reynolds number as

$$\theta_n = \frac{2^{\frac{5}{8}} \hat{c}_1^{\frac{3}{4}}}{\pi^{\frac{5}{4}}} R_{\delta^*}^{-\frac{5}{4}}.$$

We now proceed to the second order eigenrelations, and note that

$$D = 1 - \frac{\hat{\xi}_0}{\hat{c}_1}, \quad \hat{D} = \frac{4i}{3} + \frac{2\hat{\Omega}_1 \text{Ai}_0}{\kappa i^{\frac{2}{3}} \hat{\theta}_1^{\frac{2}{3}}}.$$

We require that Ω_2 and θ_2 are real, and thus we find that

$$\theta_2 = 1.55721t^{-\frac{3}{4}}.$$

In order to determine the logarithmic wave characteristics we note that $\lambda_3 = -1/(12\sqrt{\pi t^3})$, and we find that

$$\theta_{3L} = -0.56748t^{-\frac{7}{8}}.$$

Subsequent terms in the expansion are effected by the growth of the layer, hence we defer their discussion to the next section. We may combine the expressions for θ_1 , θ_2 and θ_{3L} in order to predict the neutral wavenumber θ_n in terms of the local boundary layer Reynolds number. We find that

$$\theta_n = 1.0789 R_{\delta^*}^{-\frac{5}{4}} \left(1 + 3.37079 R_{\delta^*}^{-\frac{1}{4}} - 5.70354 R_{\delta^*}^{-\frac{1}{2}} \ln R_{\delta^*} + \dots \right),$$

this value is compared with the neutral value found from Orr-Sommerfeld theory, the results are displayed in figure 4. In order to predict the upper branch we would need to consider a more complicated structure in which the critical layer is distinct from the viscous wall layer.

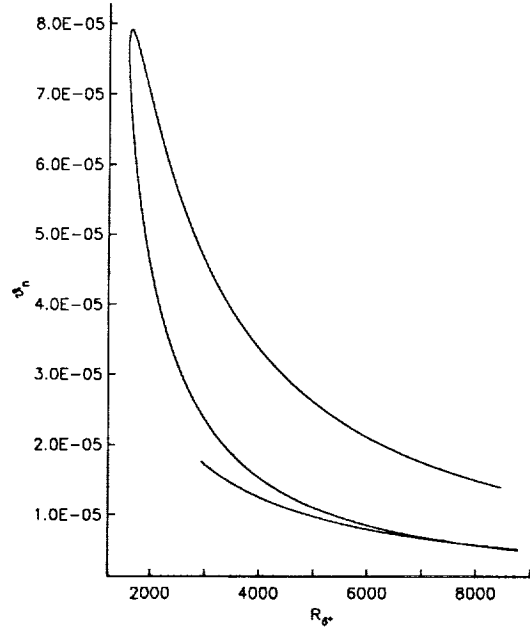


Figure 4: Comparison of neutral wavelength, Orr-Sommerfeld and triple deck

The agreement is reasonable, but it should be noted that analysis relies on different powers of the Reynolds numbers being distinct, however at the values of R_{δ^*} displayed in figure 4, $R_{\delta^*}^{-\frac{1}{2}} \ln R_{\delta^*}$ and $R_{\delta^*}^{-\frac{1}{4}}$ are not totally distinct. However, the triple deck theory does appear to capture the lower branches characteristics.

§5.2 Unsteady effects on triple deck stability

In this section we discuss the effects that the boundary layer's growth has on the stability of the Tollmien-Schlichting waves. As has been mentioned several times earlier, the evolution of the layer changes the characteristics at one order higher than in the spatial case of Smith (1979 a), thus it is perhaps even more crucial in these temporal cases that the layers evolution is included in the analysis. This is to be expected, since during one spatial wavelength of the wave the boundary does not evolve (spatially) as much as it would during one period (temporally). The leading order effect of the boundary layer's growth occurs in the eigenrelation (3.8 c), which determines Ω_3 and θ_3 .

We consider the total temporal derivative of a flow quantity Φ , and define its growth rate as,

$$\sigma = \frac{1}{\Phi} \frac{\partial \Phi}{\partial t}.$$

We require that the quantity σ is purely imaginary, which for a problem which does not exhibit any temporal changes translates to ensuring that the Ω 's are real. However, we know that $\partial_{\bar{t}} \rightarrow \partial_t + \epsilon^{-2} \partial_T$, so that

$$\sigma = \epsilon^{-2} \left[-i\Omega_1 - \epsilon i\Omega_2 - \epsilon^2 \ln \epsilon \Omega_{3L} - \epsilon \left(-\frac{\Phi_t}{\Phi} + i\Omega_3 \right) + \dots \right].$$

Hence in order for σ to remain totally imaginary (to the order considered), we require

$$\text{Imag}(\Omega_3) = -\text{Real}\left(\frac{\Phi_t}{\Phi}\right).$$

Using this condition together with eigenrelation (3.8 c), we can determine θ_3 and $[\text{Real}(\Omega_3) - \text{Imag}(\frac{\Phi_t}{\Phi})]$.

It is not clear as to which quantity should be used in order to represent the disturbance's growth. In Smith (1979 a) the wall pressure was used, and the results which were obtained allowed enhanced agreement with the experiments of Ross *et al* (1970). Smith noted that this implied that the boundary layer's growth had a stabilising effect, however if other quantities were chosen to represent the disturbance, the conclusions could be the opposite. In Gaster (1974) and Eagles & Weissman (1975), it was noted that the difficulty is in defining the instability in some absolute sense, however this method should allow good agreement with a particular experiment (in which the disturbance is measured using a certain quantity). We are not aware of any experimental data concerning the susceptibility of a Rayleigh layer to Tollmien-Schlichting waves. If, for example we take the wall layer normal velocity (V_1) to be representative of the disturbance, then the layers growth is seen to have a destabilising effect.

The reason for the enhanced agreement with experiment seen in Smith (1979 a), is likely to have its origin in the size of the correction, it is now of order $R_e^{-\frac{3}{8}}$ which is larger than that of Bouthier (1973) and Gaster's (1974) (order $R_e^{-\frac{1}{2}}$) successive approximation technique. As mentioned previously our correction is of order $R_e^{-\frac{1}{4}}$ which represents an even larger change, so we expect even better agreement between the theory and experimental results.

§6 Discussion of the weakly nonlinear stability.

The relationship between the equilibrium amplitude solutions of (4.5) and those of the unsteady flow amplitude solutions of (4.7) is now considered. The latter as previously stated corresponds to a disturbance of size $h \sim R_e^{-\frac{1}{16}}$. The former is for larger disturbances where $h \gg R_e^{-\frac{1}{16}}$. Thus we are required to solve,

$$\frac{dB}{dt} = i\tilde{\Omega}B + \sigma_2 B|B|^2, \quad (6.1)$$

and,

$$\frac{dB}{dt} = iq_1 \left(-t_n + H^{-4}\tilde{T} \right) B + \sigma_2 B|B|^2, \quad (6.2)$$

where $B = B_1$ is the lower deck amplitude function as in (4.3 c). The first equation (6.1) has the stable equilibrium solution,

$$|B|^2 = \frac{\tilde{\Omega}_i}{\sigma_{2r}}, \quad (6.3)$$

which bifurcates supercritically at $\tilde{\Omega}_i = 0$ from the zero solution for increasing Reynolds number, where σ_{2r} is always less than zero as shown in table 1. Increasing the Reynolds number corresponds to progressing through time, thus increasing t_0 . The zero solution is unstable beyond $\tilde{\Omega}$ equals zero, so any perturbation to this tends to the solution (6.3). Now note that sufficiently close to the neutral time t_n (specifically within $h \sim R_e^{-\frac{1}{16}}$), the appropriate equation is (6.2). Drawing on the conclusions of Hall & Smith (1984) we let $B = \rho \exp(i\phi)$, where ρ and ϕ are real functions of \tilde{T} . It was shown that the large \tilde{T} behaviour of ρ is given by,

$$\rho^2 \sim \frac{q_i H^{-4} \tilde{T}}{\sigma_{2r}} \quad \left(\text{as } \tilde{T} \rightarrow \infty \right).$$

It may be seen from the numerical results of section 5 that q_i and σ_{2r} are both always negative. This equation implies that the large \tilde{T} behaviour may be inferred from (6.2) neglecting the dB/dt term. This equilibrium solution corresponds to letting $\tilde{\Omega} \rightarrow 0$ in equation (6.1), which corresponds to the disturbance size being decreased. Thus the interval over which unsteady terms induce a contribution to the amplitude equation is crucial. In this interval a small disturbance ($h \sim R_e^{-\frac{1}{16}}$), develops smoothly into a finite amplitude Tollmien-Schlichting wave. Only 'later' (that is for relatively large time) does the amplitude tend towards the amplitude predicted by (6.1).

§7 Conclusions

We have shown that in theory a Rayleigh layer is unstable to Tollmien-Schlichting waves. An Orr-Sommerfeld calculation is performed and it is found that beyond a certain Reynolds number, disturbances with fixed spatial wavenumber comparable to the displacement thickness are temporally unstable. As the Reynolds number increases further the structure of the neutral modes can be described by a triple deck method, and eigenrelations are calculated and used to determine the wave characteristics. A multiple scales technique is used to determine the effect of the growth of the layer on the waves' stabilities. In summary it is found that this depends on which quantity is chosen to represent the waves intensity. A Stuart-Watson amplitude equation was derived in the neighbourhood of the neutral time, and the waves were found to bifurcate supercritically from the zero solution. An amplitude is found that represents a crucial stage in the development of the wave, this phase allows the wave to evolve smoothly from its linear stage to its equilibrium amplitude.

Although this work in this paper represents fairly minor modifications to various earlier papers; the methods are applied to an entirely different physical problem. It will be of interest to determine whether the upper branch modes can also be compared with the Orr-Sommerfeld calculation. The calculation will be solved using the methods discussed in Bodonyi & Smith (1981), wherein the effect of non-parallelism on upper branch Tollmien-Schlichting waves in a Blasius boundary layer is discussed. The upper branch problem is far more complex, since the critical layer is distinct from the wall layer. Within the context of this spatially homogeneous problem it will be interesting to find what acoustic radiation may be generated by these waves, and whether the waves on the upper branch or the lower branch of the neutral curve produce the most sound. The noise can be calculated using methods similar to those discussed in Tam & Morris (1980).

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Appendix

The double integral σ_2 involved in the weakly nonlinear calculation occurring in equation (4.5), is given by

$$\sigma_2 = \frac{1}{\theta(\theta^2 + \gamma^2)^{\frac{1}{2}} \text{Ai}(\xi_0) - \text{Ai}''(\xi_0)} \left(\frac{(\theta^2 + \gamma^2)^{\frac{1}{2}}}{i\lambda} \int_{\xi_0}^{\infty} LG(q) dq + \frac{\partial}{\partial \xi} LG(\xi_0) \right), \quad (\text{A.1})$$

where the function $G(\xi)$ is given by,

$$\begin{aligned} G(\xi) = & \bar{\Delta}^{-\frac{1}{3}} \Delta^{-\frac{1}{3}} \left(\sigma_{(a)} \left(\frac{1}{2} \int_{\xi_0}^{\xi} \bar{\text{Ai}}(q) dq \text{Ai}(\hat{\xi}_0) + \frac{1}{2} \int_{\xi_0}^{\xi} \int_{\xi_0}^{\xi} \bar{\text{Ai}}(q) dq \frac{\partial \text{Ai}(\hat{\xi})}{\partial \xi} \right. \right. \\ & \left. \int_{\xi_0}^{\xi} \text{Ai}(q) dq \bar{\text{Ai}}(\xi) + \int_{\xi}^{\xi} \int_{\xi}^{\xi} \text{Ai}(q) dq \frac{\partial \bar{\text{Ai}}}{\partial \xi} \right) \\ & + \left(\frac{1}{2^{\frac{5}{3}}} \int_{\xi_0}^{\xi} \bar{\text{Ai}}(q) dq L_2(F(\xi)) + \frac{1}{2^{\frac{5}{3}}} \int_{\xi_0}^{\xi} \int_{\xi_0}^{\xi} \bar{\text{Ai}}(q) \frac{\partial}{\partial \xi} L_2(F(\xi)) \right. \\ & \left. \frac{1}{2^{\frac{2}{3}}} \int_{\xi_0}^{\xi} L_2(F(q)) dq \bar{\text{Ai}}(\xi) + \frac{1}{2^{\frac{2}{3}}} \int_{\xi_0}^{\xi} \int_{\xi_0}^{\xi} L_2(F(q)) dq \frac{\partial \bar{\text{Ai}}}{\partial \xi} \right) \Bigg) \\ & - \Delta^{-\frac{2}{3}} \left(\int_{\xi_0}^{\xi} \int_{\xi_0}^{\xi} \text{Ai}(\xi) dq d\xi \frac{\partial^2 H}{\partial \xi^2} + \text{Ai}(\xi) H(\xi) \right), \\ \sigma_{(a)} = & \frac{2^{\frac{4}{3}} (\theta^2 + \gamma^2)^{\frac{1}{2}} \frac{\theta}{i\lambda} \int_{\xi_0}^{\infty} L_2(F(\xi)) d\xi - \Delta^{\frac{2}{3}} \frac{\partial}{\partial \xi} L_2(F(\hat{\xi}_0))}{4(\theta^2 + \gamma^2)^{\frac{1}{2}} \frac{\theta}{i\lambda} \int_{\xi_0}^{\infty} \text{Ai}(q) dq + 2^{\frac{2}{3}} \Delta^{\frac{2}{3}} \text{Ai}'(\hat{\xi}_0)}, \end{aligned}$$

and the form of $F(\xi)$ included in the integral $\sigma_{(a)}$ is given by,

$$F(\xi) = \frac{2}{\Delta^{\frac{2}{3}}} \left(\int_{\xi_0}^{\xi} \text{Ai}(q) dq \text{Ai}(\xi) - \int_{\xi_0}^{\xi} \int_{\xi_0}^{\xi} \text{Ai}(\xi) d\xi d\xi \frac{\partial \text{Ai}}{\partial \xi} \right),$$

the function $H(\xi)$ is given by,

$$\begin{aligned} \frac{\partial^3 H(\xi)}{\partial \xi^3} = \Delta^{-\frac{2}{3}} & \left(i^{\frac{2}{3}} \int_{\xi_0}^{\xi} \int_{\xi_0}^q \text{Ai}(q) dq d\xi \frac{\partial \bar{\text{Ai}}}{\partial \xi} - i^{\frac{2}{3}} \int_{\xi_0}^{\xi} \int_{\xi_0}^q \bar{\text{Ai}}(q) dq d\xi \frac{\partial \text{Ai}}{\partial \xi} \right. \\ & \left. + \frac{\partial}{\partial \xi} (\text{Ai}(\xi) \bar{\text{Ai}}(\xi)) \right), \end{aligned}$$

where the operators L and L_2 are defined as,

$$L(\phi) = \frac{\partial^2 \phi}{\partial \xi^2} - \xi \frac{\partial \phi}{\partial \xi},$$

$$L_2(\phi) = \frac{\partial^2 \phi}{\partial \xi^2} - 2\xi \frac{\partial \phi}{\partial \xi}.$$

The terms $c.t.\lambda_3$ occurring in equation (3.8), that is those due to the second order basic flow terms, are given by

$$c.t.\lambda_3 = \frac{\theta_1^2}{\lambda} \Delta^{-\frac{2}{3}} \int_{\xi_0}^{\infty} f(\xi) - i \Delta^{\frac{2}{3}} \frac{\partial f}{\partial \xi},$$

where $f(\xi)$ as given in section 3.5.

The terms due to non-conventional analysis in equation (3.8), are

$$\begin{aligned} n.t. = \frac{1}{B_1} \frac{\partial B_1}{\partial t} & \left(\text{Ai}'_0 \frac{\Delta^{-\frac{1}{3}} \theta_1}{\lambda^2 i} - i \text{Ai}_0''' \right) + \frac{1}{\Omega_1} \frac{\partial \Omega_1}{\partial t} \left(-\Delta^{-\frac{1}{3}} \text{Ai}_0'' \frac{\theta_1 \Omega_1}{\lambda^2} - \Omega_1 \text{Ai}_0^{(iv)} \right) \\ & + \frac{1}{\lambda} \frac{\partial \lambda}{\partial t} \left(-\frac{\theta_1^2}{\lambda} \text{Ai}'_0 + \Omega_1 \text{Ai}_0'' \Delta^{-\frac{1}{3}} \frac{\theta_1}{\lambda^2} - i \Delta^{\frac{2}{3}} \text{Ai}_0''' + \text{Ai}_0^{(iv)} \left(\frac{\Omega_1}{3} - \frac{1}{15} \frac{\theta_1^2}{\lambda} \Delta^{-\frac{2}{3}} \right) + \frac{i}{15} \Delta^{\frac{2}{3}} \text{Ai}_0^{(vi)} \right). \end{aligned} \quad (A.2)$$

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